Conditional Probability

INFO-2301: Quantitative Reasoning 2

Michael Paul and Jordan Boyd-Graber

SLIDES ADAPTED FROM DAVE BLEI AND LAUREN HANNAH

Context

- Data science is often worried about “if-then” questions
  - If my e-mail looks like this, is it spam?
  - If I buy this stock, will my portfolio improve?
- Since data science uses the language of probabilities, we need *conditional* probabilities (continuing probability intro)
- Also need to **combine** distributions

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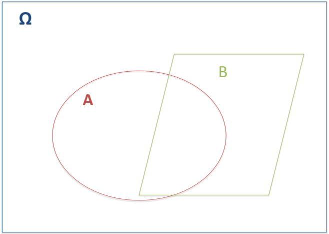

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Independence (Reminder)

Random variables X and Y are independent if and only if $P(X=x, Y=y) = P(X=x)P(Y=y)$. How does this interact with conditional probabilities?

Conditional probabilities equal unconditional probabilities with independence:

- $P(X=x \mid Y) = P(X=x)$
- *Knowing Y tells us nothing about X*

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$$= \frac{1}{9}$$

Combining Distributions

- Somtimes distributions you have aren't what you need
  - Conditional $\longrightarrow$ joint (chain)
  - Reverse conditional direction (Bayes')

The chain rule

- The definition of conditional probability lets us derive the *chain rule*, which let's us define the joint distribution as a product of conditionals:

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- For example, let Y be a disease and X be a symptom. We may know $P(X|Y)$ and $P(Y)$ from data. Use the chain rule to obtain the probability of having the disease and the symptom.
- In general, for any set of N variables

$$P(X_1, \ldots, X_N) = \prod_{n=1}^{N} P(X_n|X_1, \ldots, X_{n-1})$$

Bayes' Rule

What is the relationship between $P(A|B)$ and $P(B|A)$?

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

1. Start with $P(A|B)$
2. Change outcome space from B to Ω
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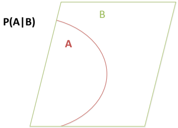

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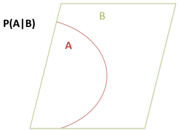

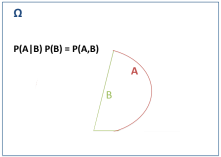

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