Probability Basics
Part 2: Probability Operations

INFO-1301, Quantitative Reasoning 1
University of Colorado Boulder

March 3, 2017

Prof. Michael Paul

Operations

Last month we learned about different mathematical *operations* for sets and booleans:

Sets:	**Booleans:**
Intersection	AND
Union	OR
Complement	NOT

These operations can also be used to compute probabilities for random variables

Example

Consider the probability of different outcomes from the roll of a die

$P(X=1) = 1/6$
$P(X=2) = 1/6$
$P(X=3) = 1/6$
$P(X=4) = 1/6$
$P(X=5) = 1/6$
$P(X=6) = 1/6$

Distribution over 6 possible outcomes

A distribution where all outcomes are equally likely is a **uniform** distribution

Disjunctions

If two or more outcomes cannot all be true at once, they are called **disjoint** or **mutually exclusive**

Die roll outcomes are disjoint

- A die cannot land a 3 and also a 4

Complements

The **complement** of an outcome is the set of all other outcomes in the sample space

- This is the same as the set complement operation that you learned about before

Remember:

Sample space is the *domain* of the random variable, which is a *set* of the possible outcomes

When discussing complements, we assume the outcomes are disjoint

AND

The probability that multiple outcomes are true can be described with an AND expression

$P(X=3 \text{ AND } X=4) = 0$

If the outcomes are disjoint, the probability of the AND of multiple outcomes will always be 0

AND

The probability that multiple outcomes are true can be described with an AND expression

Harder example: two dice
X is outcome of first die; Y is outcome of second

P(X=3 AND Y=4) = 1/36
P(X=4 AND Y=3) = 1/36
P(X=4 AND Y=4) = 1/36
…

OR

The probability that *any* outcome is true can be described with an OR expression

$P(X{=}3 \text{ OR } X{=}4) =$ 2/6

Addition rule:
If outcomes are disjoint, the probability that any of them are true is the *sum* of their individual probabilities

OR

The probability that *any* outcome is true can be described with an OR expression

$$
\begin{aligned}
P(X > 3) &= P(X{=}4 \text{ OR } X{=}5 \text{ OR } X{=}6) \\
&= P(X{=}4) + P(X{=}5) + P(X{=}6) \\
&= 1/6 + 1/6 + 1/6 \\
&= 1/2
\end{aligned}
$$

OR

What if the outcomes aren't disjoint?

Harder example: two dice
X is outcome of first die; Y is outcome of second

$P(X{=}3 \text{ OR } Y{=}4) = ?$

$P(X{=}3) + P(Y{=}4)$ isn't quite right:
the outcome $X{=}3$ AND $Y{=}4$ is counted twice

OR

What if the outcomes aren't disjoint?

Harder example: two dice
X is outcome of first die; *Y* is outcome of second

$P(X=3 \text{ OR } Y=4) =$

$P(X=3) + P(Y=4)$

$- P(X=3 \text{ AND } Y=4)$ ⟵ Subtract out the AND which is double counted

OR

What if the outcomes aren't disjoint?

General addition rule: (for two outcomes)
The probability that either outcome is true is the *sum* of their individual probabilities, *minus* the probability that they are both true

- i.e., P(X OR Y) = P(X) + P(Y) – P(X AND Y)

Similar to calculating the cardinality of a set union:
$|A \cup B| = |A| + |B| - |A \cap B|$

NOT

The probability that an outcome is *not* true is the probability of any other outcome in the sample space

$P(X \text{ is NOT } 3)$

$= P(X \neq 3)$

$= P(X=1 \text{ OR } X=2 \text{ OR } X=4 \text{ OR } X=5 \text{ OR } X=6)$

$= 5/6$

$= 1 - P(X=3)$

NOT

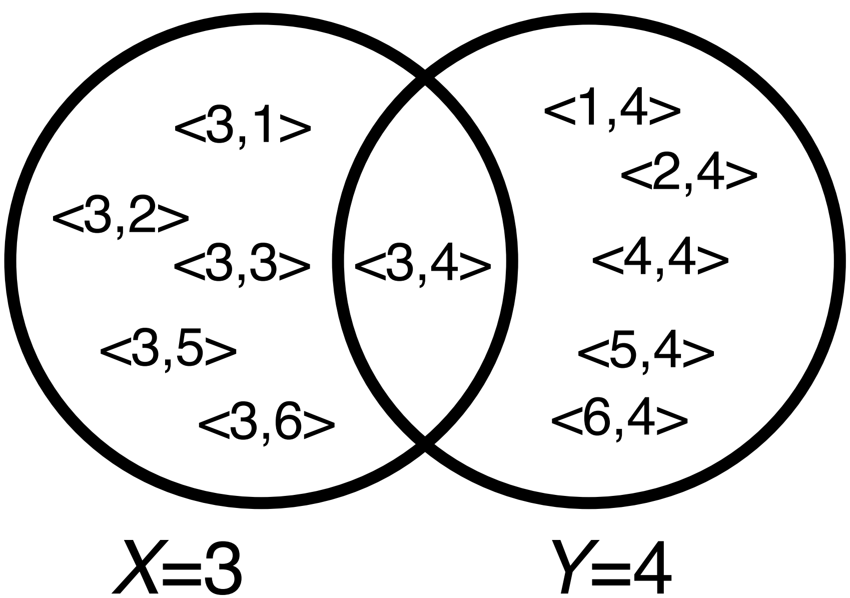
The probability of the complement of an outcome is always 1 minus the probability of the outcome

$$P(\text{NOT } X) = 1 - P(X)$$

Venn Diagrams

If you have multiple outcomes you can draw the relationships between them as a Venn diagram

- That is, draw the sets of the outcomes that correspond to what you are calculating



The *intersection* is X=3 AND Y=4
The *union* is X=3 OR Y=4

The *complement* of X=3 is X≠3

- We assume the universal set is all possible dice rolls

Independence

Two random variables are **independent** if knowing the outcome of one does not change the probability of the other

Independent:

- Probability of getting heads twice when you flip two coins

Not independent:

- Probability of snow this weekend and the probability of high traffic on I-70

Independence

If two random variables are independent, then the probability of two outcomes is simply the product of the two outcomes individually.

If X and Y are independent:

$P(X = a \text{ AND } Y = b) = P(X = a) \times P(Y = b)$

This idea extends to more than two variables:

$P(X = a \text{ AND } Y = b \text{ AND } Z = c) = P(X{=}a) \times P(Y{=}b) \times P(Z{=}c)$

Independence

What's the probability of flipping 5 heads in a row?

$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = 1 / 32$

Same answer we got on the board last time.

2^5 possible permutations of coin flips, and only one of them has 5 heads.